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Periodic Solutions of Nonlinear Delay Equations

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Existence and uniqueness of 2π-periodic solutions of $d^jx(t)/dt^j + \operatorname{grad} G(x(t-\tau)) = e(t, x(t), x(t-\tau))$ $(j = 1, 2)$, where $x(t)$ is in $\mathbb{R}^n$ and $e(t, u, v)$ is a given vector function, 2π-periodic in t, are shown under conditions on the spectrum of the Hessian of G. The equation is studied using a fixed point theorem in the space $L^2(0, 2\pi)$. One feature of this approach is that no relationship between the delay and the period is necessary.

1. INTRODUCTION AND PRELIMINARIES

The ordinary differential equation

$$\frac{d^2x}{dt^2} + \operatorname{grad} G(x) = e(t) \tag{1}$$

has recently received much attention. In [5] Lazer and Sánchez prove the following theorem.

THEOREM [Lazer–Sánchez]. *Let $e \in C(\mathbb{R}, \mathbb{R}^n)$ be 2π-periodic. If G is in $C^2(\mathbb{R}^n, \mathbb{R})$ and if there exists an integer n and numbers p and q such that $n^2 < p \leqslant q < (n+1)^2$, and if*

$$pI \leqslant (\partial^2 G(a))/(\partial x_i \partial x_k) \leqslant qI$$

for all $a \in \mathbb{R}^n$, then there exists a 2π-periodic solution of Eq. (1).

Various generalizations of this theorem have recently appeared. (See Lazer [4], Mawhin [6], Reissig [7], Ward [8], and Kannan and Locker [2].) Here we use the approach of Mawhin and Ward to extend these results to equations with delay.

In this paper, $L^2(0, 2\pi)$ will denote the Hilbert space of 2π-periodic measurable functions $x: \mathbb{R}^1 \to \mathbb{R}^n$ with norm

$$\|x\|^2 = (x, x) = (\pi)^{-1} \int_0^{2\pi} |x(t)|^2 \, dt < \infty,$$

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where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. For matrices A and B "$A \geqslant B$" is taken to mean that matrix $(A - B)$ is positive semidefinite.

2. Formulation of the Theorem

We shall consider the problem of finding 2π-periodic solutions to (2) in $L^2(0, 2\pi)$:

$$\frac{d^j x(t)}{dt^j} + \operatorname{grad} G(x(t-\tau)) = e(t, x(t), x(t-\tau)), \qquad (j = 1, 2). \qquad (2)$$

We shall make the following assumptions on the terms in (2).

Assumption A1. $e(t, u, v)$ is continuous and for all $(t, u, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$,

$$|e(t, u, v)| \leqslant E < \infty, \qquad (3)$$

and $e(t + 2\pi, u, v) = e(t, u, v)$.

Assumption A2. $G \in C^2$ $(\mathbb{R}^n, \mathbb{R})$, and there is an integer m and numbers p, q with

$$m^j I < pI \leqslant H(a) \leqslant qI \leqslant (m+1)^j I, \qquad a \in \mathbb{R}^n, \qquad (4)$$

where H is the Hessian of G, $H(a) = (\partial^2 G(a))/(\partial x_i \partial x_k)$.

Furthermore we may assume $\operatorname{grad} G(0) = 0$ since otherwise we may subtract $\operatorname{grad} G(0)$ from both sides of (2) and treat an equation with this property. Since this transformation leaves A1 and A2 unchanged there is no loss of generality in making this assumption.

Then the following holds.

Theorem. *Equation* (2), *with* $j = 1$ *or* 2, *admits at least one* 2π*-periodic solution in* $L^2(0, 2\pi)$ *under assumptions* A1 *and* A2. *Furthermore when* $e(t, u, v) = e(t) \in L^2(0, 2\pi)$ *that solution is unique and depends continuously upon* e.

For the rest of this paper we shall use the following notation. L_v^j will denote the linear operator attached to (2), as follows:

L_v^j: $D^j \to L^2(0, 2\pi)$ has domain,
$D^j = \{x \in L^2(0, 2\pi)$: $d^k x/dt^k$ is absolutely continuous for $k = 0, j-1$, $d^j x/dt^j \in L^2(0, 2\pi)$, and $d^k x(0)/dt^k = d^k x(2\pi)/dt^k$, $k = 0, j-1\}$,

and values

$$(L_\nu^j x)(t) = \frac{d^j x(t)}{dt^j} + \nu x(t-\tau). \tag{5}$$

Define the (nonlinear) operators

$$(Mx)(t) = e(t, x(t), x(t-\tau)), \tag{6}$$

and

$$(N_\nu x)(t) = \nu x(t-\tau) - \text{grad}\, G(x(t-\tau)). \tag{7}$$

In [3] Krasnosel' skii shows that if $f(t, u)$ satisfies the Carathéodory conditions and if whenever $u(t)$ is in L^{p_1}, $v(t) = f(t, u(t)) \in L^{p_2}$ $(p_1, p_2 \geqslant 1)$, then the operator $u(t) \to f(t, u(t))$ is continuous and bounded from L^{p_1} to L^{p_2}. This result with assumptions A1 and A2 ensures that M and N_ν are continuous and bounded maps of $L^2(0, 2\pi)$ into $L^2(0, 2\pi)$. Equation (2) is rewritten as

$$\frac{d^j x(t)}{dt^j} + \nu x(t-\tau)$$
$$= e(t, x(t), x(t-\tau)) + \nu x(t-\tau) - \text{grad}\, G(x(t-\tau))$$

and shall henceforth be studied as the fixed point equation

$$x = (L_\nu^j)^{-1}\,(Mx + N_\nu x) \equiv T_\nu x. \tag{8}$$

3. Proof of the Theorem

We shall need some results concerning the linear operator L_ν^j. If $f(t)$ is a given function in $L^2(0, 2\pi)$

$$f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt), \tag{9}$$

then the unique solution to $L_\nu^j x = f$ in D^j is given by (see El'Sgol'Ts [1])

$$x(t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} (C_n \cos nt + D_n \sin nt), \tag{10}$$

where, for $j = 1$ and $\nu \neq 0, 1, 2, \ldots$,

$$C_0 = \frac{\alpha_0}{\nu}, \qquad C_n = \frac{U_n \alpha_n - V_n \beta_n}{U_n^2 + V_n^2}, \qquad D_n = \frac{U_n \beta_n + V_n \alpha_n}{U_n^2 + V_n^2}$$

with $U_n = \nu \cos n\tau$ and $V_n = n - \nu \sin n\tau$, and for $j = 2$ and $\nu \neq 0, 1^2, 2^2, \ldots$,

$$C_0 = \frac{\alpha_0}{\nu}, \qquad C_n = \frac{P_n \alpha_n - Q_n \beta_n}{P_n^2 + Q_n^2}, \qquad D_n = \frac{P_n \beta_n + Q_n \alpha_n}{P_n^2 + Q_n^2}$$

with $P_n = \nu \cos n\tau - n^2$ and $Q_n = -\nu \sin n\tau$.

First we note that the inequality

$$P_n^2 + Q_n^2 = (\nu - n^2)^2 + 2\nu n^2(1 - \cos n\tau) \geqslant (\nu - n^2)^2 \tag{11}$$

yields a bound on $\|(L_\nu^2)^{-1}\|$. Indeed,

$$\begin{aligned} \|x\|^2 = \|(L_\nu^2)^{-1} f\|^2 &= \frac{|C|}{2} \\ &+ \sum_{n=1}^{\infty} \left(\left| \frac{P_n \alpha_n - Q_n \beta_n}{P_n^2 + Q_n^2} \right| \right)^2 + \left(\left| \frac{P_n \beta_n + Q_n \alpha_n}{P_n^2 + Q_n^2} \right| \right)^2 \\ &\leqslant \frac{1}{2} \left| \frac{\alpha_0}{\nu} \right|^2 + \sum_{n=1}^{\infty} \frac{(|\alpha_n|^2 + |\beta_n|^2)(P_n^2 + Q_n^2)}{(P_n^2 + Q_n^2)^2} \\ &\leqslant d_2^{-2} \left\{ \frac{|\alpha_0|^2}{2} + \sum_{n=1}^{\infty} |\alpha_n|^2 + |\beta_n|^2 \right\} = d_2^{-2} \|f\|^2, \end{aligned}$$

where $d_2 = \min \{|\nu - n^2| : n = 0, 1, \ldots\} > 0$. Thus $\|(L_\nu^2)^{-1}\| \leqslant d_2^{-1}$. In a similar manner the inequality

$$U_n^2 + V_n^2 \geqslant (\nu - n)^2 \tag{12}$$

gives $\|(L_\nu^1)^{-1}\| \leqslant d_1^{-1}$ with $d_1 = \min\{|\nu - n| : n = 0, 1, 2, \ldots\}$. To summarize:

LEMMA 1. *Given $f \in L^2(0, 2\pi)$ and $\nu \neq n^j$ $(n = 0, 1, \ldots)$ there is a unique solution to $L_\nu^j x = f$, given by* (10), *in D^j. Furthermore*

$$\|x\| = \|(L_\nu^j)^{-1} f\| \leqslant d_j^{-1} \|f\|,$$

where $d_j = \min\{|\nu - n^j| : n = 0, 1, \ldots\} > 0$.

We shall also need

LEMMA 2. *The operator $(L_\nu^j)^{-1}$ is compact on $L^2(0, 2\pi)$.*

Proof of Lemma 2. $(L_\nu^j)^{-1}$ is shown to be the limit of operators with finite dimensional range. For f as in (9) define $S_k f = C_0/2 + \sum_{n=1}^{k} (C_n \cos nt + D_n \sin nt)$. For $k > \nu$ inequalities (11) and (12) give

$$\|(L_\nu^j)^{-1}f - S_k f\|^2 = \sum_{n=k+1}^{\infty} (|C_n|^2 + |D_n|^2)$$
$$\leqslant \frac{1}{(\nu-(k+1))^2}\|f\|^2, \qquad \text{when} \quad j=1,$$
$$\leqslant \frac{1}{(\nu-(k+1)^2)^2}\|f\|^2, \qquad \text{when} \quad j=2.$$

In either case $\|(L_\nu^j)^{-1} f - S_k f\|/\|f\|$ goes to zero as $k \to \infty$ and $(L_\nu^j)^{-1}$ is compact.

For the proof of the existence portion of the theorem we shall use the

SCHAUDER FIXED POINT THEOREM. *Let C be a compact convex set in a Banach space and T a continuous map of C into itself. Then T has a fixed point in C, that is, $Tx = x$ for some x in C.*

The operators M and N_ν are continuous and $(L_\nu^j)^{-1}$ is compact. If $T_\nu = (L_\nu^j)^{-1}(M + N_\nu)$ maps some closed ball $B_R(0) = \{x \in L^2(0, 2\pi)$: $\|x\| \leqslant R\}$ into itself, then we can apply the fixed point theorem to C, C being the closed convex hull of $T_\nu(B_R(0))$, to obtain existence of a solution of (8).

Assumption A2 and the mean value theorem yield

$$\begin{aligned}\|N_\nu x - N_\nu y\| &\leqslant \sup\{\|\nu I - H(a)\| : a \in \mathbb{R}^n\}\,\|x-y\| \\ &\leqslant \max\{|\nu-p|, |\nu-q|\}\,\|x-y\|.\end{aligned} \tag{13}$$

Using (13), in the case $y=0$, and the estimates on $\|(L_\nu^j)^{-1}\|$ gives

$$\begin{aligned}\|T_\nu x\| &\leqslant \|(L_\nu^j)^{-1}\|\,(\|Mx\| + \|N_\nu x\|) \\ &\leqslant d_j^{-1}\,2^{1/2}E + d_j^{-1}\max\{|\nu-p|, |\nu-q|\}\,\|x\|,\end{aligned}$$

where we have used A1 for $\|Mx\| \leqslant 2^{1/2}E$ and grad $G(0) = 0$ for $N_\nu(0) = 0$.

Here ν is chosen to make $\alpha(\nu) \equiv d_j^{-1}\max\{|\nu-p|, |\nu-q|\} < 1$, or equivalently

$$\max\{|\nu-p|, |\nu-q|\} < \min\{|\nu-n^j|, |\nu-(n+1)^j|\}.$$

Indeed, from Fig. 1, $\alpha(\nu) < 1$ for any ν satisfying $\nu_1 < \nu < \nu_2$, where

$$\nu_1 = \frac{n^j + q}{2} \qquad \text{and} \qquad \nu_2 = \frac{(n+1)^j + p}{2}.$$

With ν so chosen we now show that T_ν maps $B_k(0)$ into $B_k(0)$ for some k. If not, there is a sequence $\{y_k\}$ in $L^2(0, 2\pi)$ with $y_k \in B_k(0)$ and $\|T_\nu y_k\| > k$.

From

$$\|T_v y_k\| \leqslant d_j^{-1}\, 2^{1/2}E + \alpha(v)\, \|y_k\| \tag{14}$$

we see that $\|y_k\|$ must tend to infinity. But (14) also implies that $\|y_k\|$ must be uniformly bounded since

$$\|y_k\| \leqslant k < \|T_v y_k\| \leqslant d_j^{-1}\, 2^{1/2}E + \alpha(v)\, \|y_k\|,$$

or

$$\|y_k\| \leqslant (1-\alpha(v))^{-1}\, 2^{1/2} d_j E.$$

Thus we have proven existence.

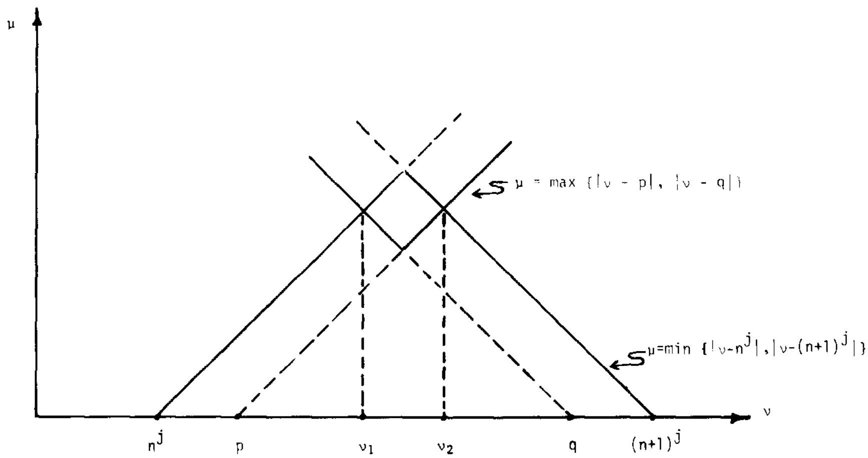

FIGURE 1

To prove uniqueness when $e = e(t)$ we note that in this case

$$\begin{aligned}\|T_v x - T_v y\| &= \|(L_v^j)^{-1}\,(N_v x - N_v y)\| \\ &\leqslant d_j^{-1} \max\{|v-p|, |v-q|\}\, \|x-y\| \\ &\leqslant \alpha(v)\, \|x-y\|.\end{aligned}$$

With v chosen, as before, to make $\alpha(v) < 1$, T_v becomes a global contraction and uniqueness follows from the contraction mapping theorem.

For continuous dependence upon $e(t)$ let x_1 and x_2 be solutions to

$$x = (L_v^j)^{-1}\,(e(t) + N_v x) \tag{15}$$

with $e=e_1$ and $e=e_2$, respectively. Subtracting x_1 from x_2 using (15) and the previous estimates yields

$$\begin{aligned}\|x_1-x_2\| &\leqslant d_j^{-1}\,\|N_r x_1 - N_r x_2\| \\ &\leqslant d_j^{-1}\,\|e_1-e_2\| + \alpha(v)\,\|x_1-x_2\|\end{aligned}$$

and hence with $\alpha(v)<1$ we have $\|x_1-x_2\| \leqslant (d_j(1-\alpha(v)))^{-1}\,\|e_1-e_2\|$, and the proof is complete.

Note. A close look at the proof shows that the conditions on $e(t,u,v)$ may be weakened to slow growth. For example, suppose

$$|e(t,u,v)| \leqslant E_1 + E_2(|u|+|v|)^{\beta}. \tag{16}$$

When $0\leqslant\beta<1$ then $\|Mx\|/\|x\|\to 0$ as $\|x\|\to\infty$ and existence holds. When $\beta=1$ and E_2 is sufficiently small existence holds also.

Further we should note that by seeking a solution defined and periodic on $\mathbb{R}^1$, rather than a solution to the initial value problem (on $\mathbb{R}^1_+$), we need impose no conditions relating the delay to the period.

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